The Collatz Conjecture over the Gaussian Integers

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Physics and Mathematics Colloquium

University of Colima

Thank you to the Colloquium Organizers!

- Alfredo Aranda

Abstract

We introduce the Collatz conjecture, as well as some elementary results. In addition, we investigate a Collatz-type function over the Gaussian and Eisenstein integers.

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(NOT TO SCALE)
Albers equal area projection
0 100 200 300 mi
0 100 200 300 km

Summer 2017

REU @ CSU Channel Islands, NSF 1359165

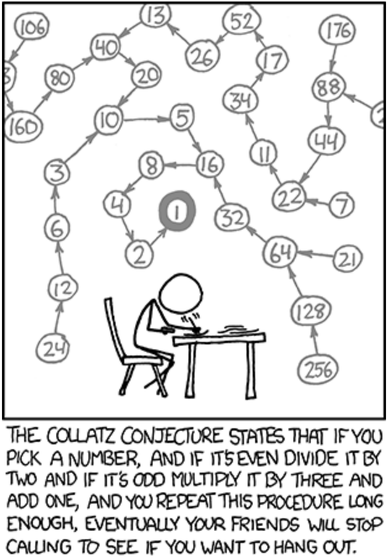
Faculty Mentors

Figure 1: Dr. Martinez-Avendano, ITAM; Dr. Martinez, CLU; AA, EIU; Dr. Sittinger, CSUCI

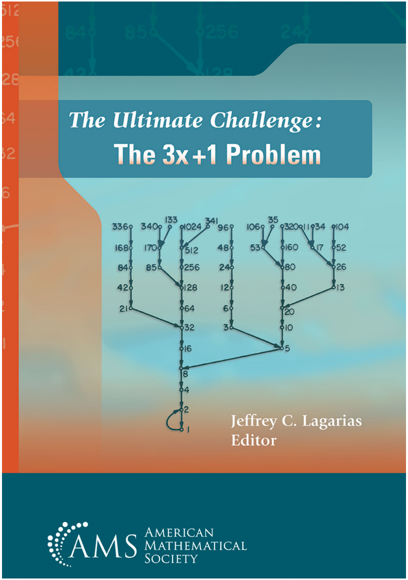
Research Group

Figure 2: E. Knutsen, CUB; R. Ceja Ayala, ASU; J. Torchinsky, Sandia

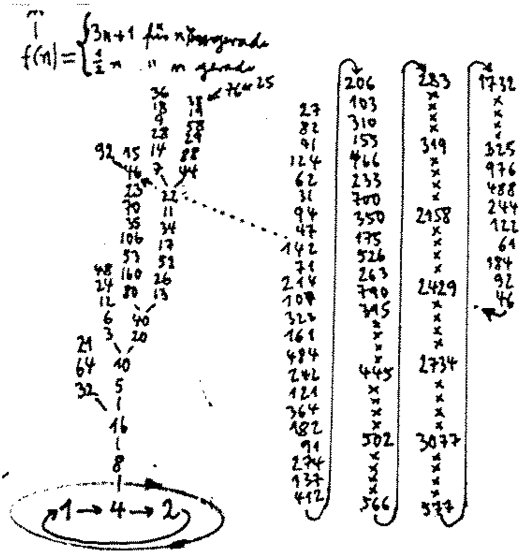
xkcd.com



Favorite Book



Hand Sketch by Lothar Collatz



- German mathematician, 1910 - 1990
- Interested in the behavior of recursive iterations of functions over the positive integers, especially in their graphical representation.
- He may have circulated the problem orally at the ICM in 1950.
- First appeared in print in 1971, written version of a lecture by Coxeter.

www.cecm.sfu.ca/organics/papers/lagarias

The Collatz Conjecture

Let $C : \mathbb{N} \to \mathbb{N}$ such that

$$C(x) = \begin{cases} 3x + 1 & \text{if } 2 \nmid x, \\ \dfrac{x}{2} & \text{if } 2 \mid x. \end{cases}$$

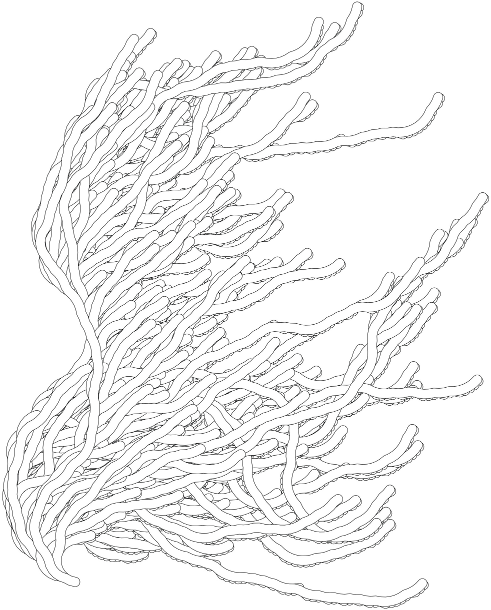
The Collatz conjecture states that the **trajectory** (hailstone sequence) of n, the sequence $\{n, C(n), C^2(n), \dots\}$, will eventually reach 1 for every positive integer n.

It has been shown to hold computationally up to ~~5×2^{60}~~ 2^{68}, Barina 2020.

"Almost all Collatz orbits [trajectories] attain almost bounded values", Tao 2019.

"This is a really dangerous problem. People become obsessed with it and it really is impossible", Lagarias.

More Collatz



This image is a sample from the colouring book Visions of the Universe by Alex Bellos and Edmund Harriss
Shared with their permission

- 2021 Collatz Conjecture Prize, mathprize.net, 120 million JPY (over 16 million MXN)
- www.bradyharanblog.com/blog/the-collatz-conjecture-in-colour
- Image is a sample from the colouring book Visions of the Universe by Bellos and Harriss
- Collatz calculator, www.dcode.fr/collatz-conjecture

The Collatz Conjecture

Why do we expect any trajectory will eventually reach 1?

Crandall (1978) presented a heuristic probabilistic argument, to come up with a stronger conjecture. He constructs a pdf and calculates an expected value.

"If one considers only the odd numbers in the sequence generated by the Collatz process, then each odd number is on average 3/4 of the previous one... This yields a heuristic argument that every Hailstone sequence should decrease in the long run, although this is not evidence against other cycles, only against divergence..."
-Lagarias

Collatz Graph

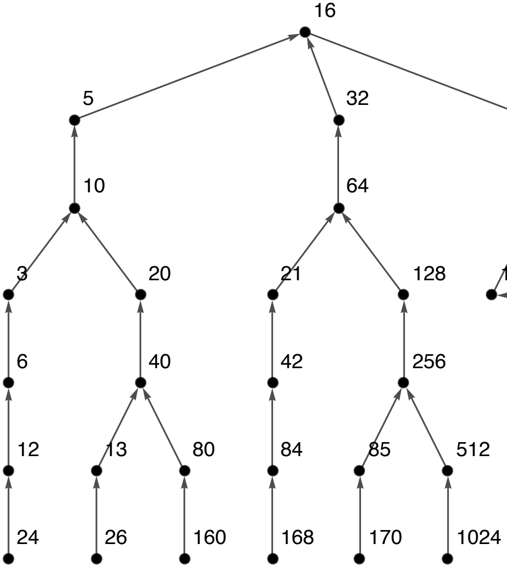
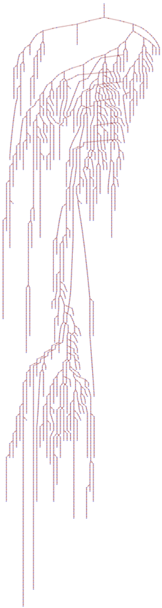


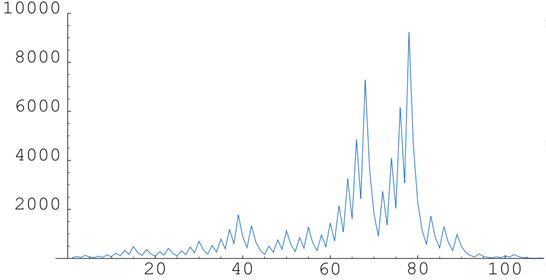
Figure 3: Trajectories in 10 steps or less

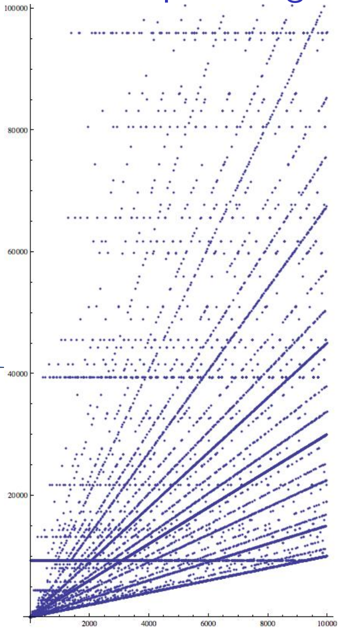
Mathematica



Collatz Trajectories

Wikipedia.org





(L) $\{27, 82, 41, ..., 9232, ..., 5, 16, 8, 4, 2, 1\}$
For $x = 27$, it took 111 steps to return to 1, with a maximum of 9232.

(R) x-axis represents starting number; y-axis represents highest number reached.

Collatz Conjecture

What about the generalized Collatz function?
What about the Collatz problem over other rings?

- Matthews and Watts (1984).
  Let $T_{d,m,r} : \mathbb{Z} \to \mathbb{Z}$ such that

$$T_{d,m,r}(x) = \frac{m_i\, x + r_i}{2} \quad \text{if } x \equiv i \mod d$$

- Kionke (2016) considers generalized Collatz mappings on free abelian groups of finite rank.
- For example the Collatz function defined over $\mathbb{Z}[\sqrt{2}]$, its trajectories were studied using computer experiments.
- Kionke's method proved the existence of divergent trajectories.

Motivation: Gaussian Moat Problem

Is it possible to "walk to infinity", using the Gaussian primes as stepping stones and taking steps of bounded-length?

We cannot walk to infinity on the real number line. We can always take steps of size n: $(n+1)!+2, (n+1)!+3, ..., (n+1)!+(n+1)$

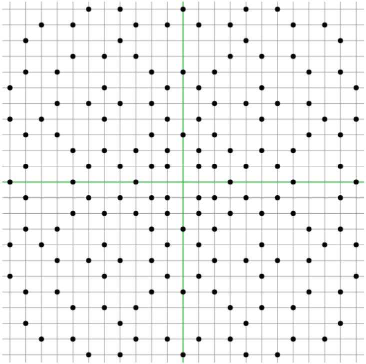


Figure 4: Black vertices are the Gaussian primes.

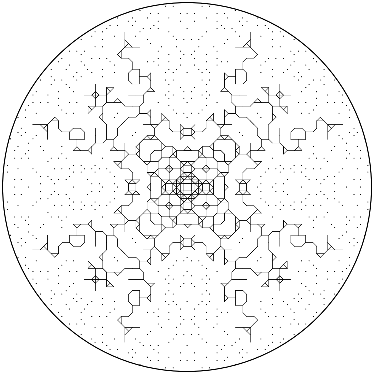


Figure 5: Reachable Gaussian primes for steps of at most 2.

Background on Gaussian Integers

Goal: Extend the Collatz Conjecture to the Gaussian Integers.

Recall some properties of the integers $\mathbb{Z}$:

- We can talk about what it means for an integer to be prime, composite, even, odd.
- We can ask if an integer is a unit, take their absolute value, apply the division algorithm.
- We know integers can be factored uniquely as a product of primes (up to associates).
- We can order the integers; in particular, the natural numbers.

Background on Gaussian Integers

We have analogous properties for the Gaussian integers, $\mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}, i^2 = -1\}$.

- If $p \equiv 3 \bmod 4$ then p is a Gaussian prime.
- If $p \equiv 1 \bmod 4$ then $a + bi$ is a Gaussian prime, where $p = (a + bi)(a - bi)$.
- The norm (square of the absolute value) $N(a + bi) = (a + bi)(a - bi) = a^2 + b^2$. The units are $\{\pm 1, \pm i\}$.
- It can be shown that Gaussian integers can also be factored uniquely as a product of primes (up to associates). For example, $2 = (1 + i)(1 - i) = i(1 - i)^2$.
- We can construct a spiral ordering of the Gaussian integers in the first quadrant.

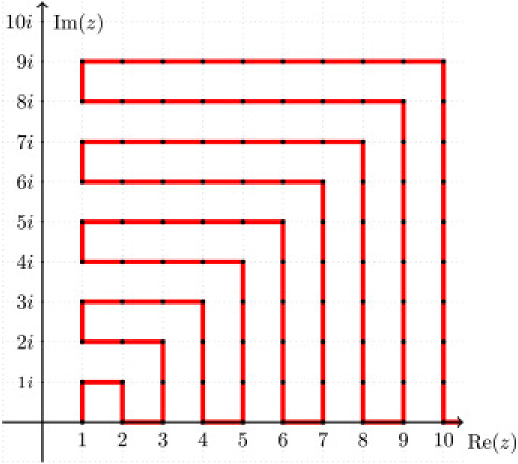
Spiral Ordering of the Gaussian Integers

How do we order the complex numbers?

2015 SUMmer REU

Mentors: Tou, Klee, Stewart

Prime Labeling of Families of Trees with the Gaussian Integers
by Klee, Lehmann, Park



Gaussian Collatz Mapping

Over $\mathbb{N}$, the first two primes are 2 and 3.

Over $\mathbb{Z}[i]$ (in first quadrant), the first two primes are $1 + i$ and $2 + i$, assuming the spiral ordering.

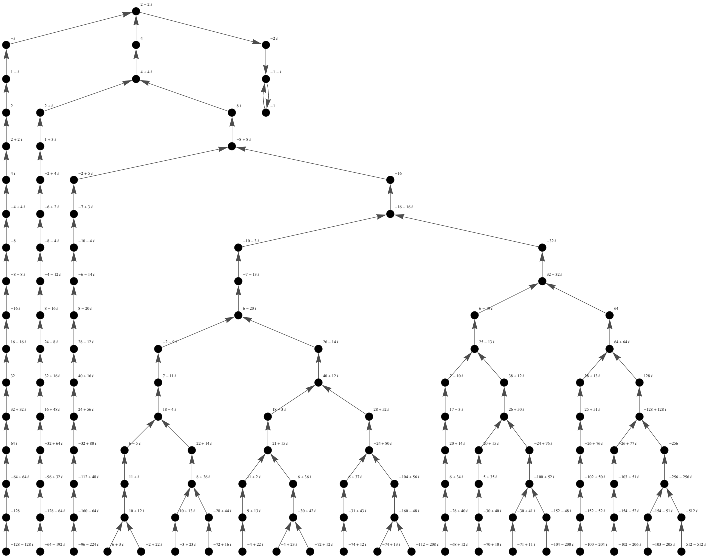
Let $C_{2+i,u} : \mathbb{Z}[i] \to \mathbb{Z}[i]$ where

$$C_{2+i,u}(z) = \begin{cases} (2+i)\,z + u, & \text{if } 1+i \nmid z \qquad z \text{ "odd"} \\ \dfrac{z}{1+i}, & \text{if } 1+i \mid z \qquad z \text{ "even"} \end{cases}$$

and u is a unit.

Analogous to the integers mod 2, there are two congruence classes mod $1 + i$, which we call even and odd Gaussian integers.

Gaussian Collatz Mapping



Trajectory of $10 - 2i$ and $6 - 7i$ using $u = 1$

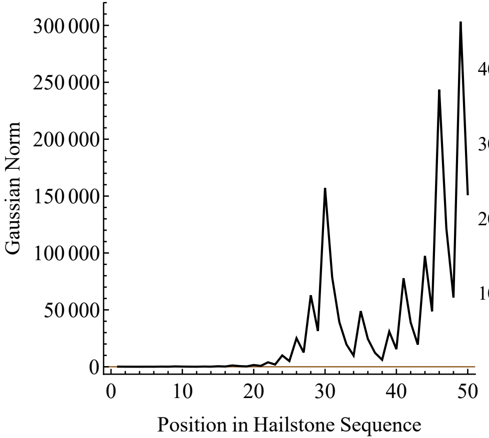


Figure 6: Does not land within 10,000 steps (up to 50 steps shown).

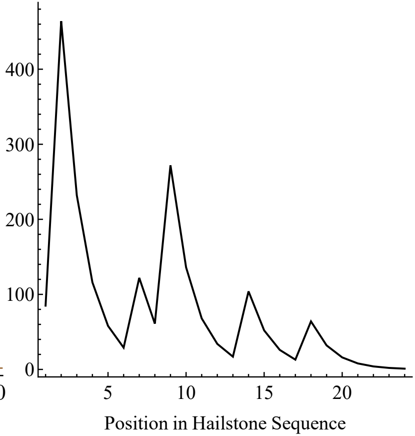


Figure 7: Lands on 1 in 24 steps.

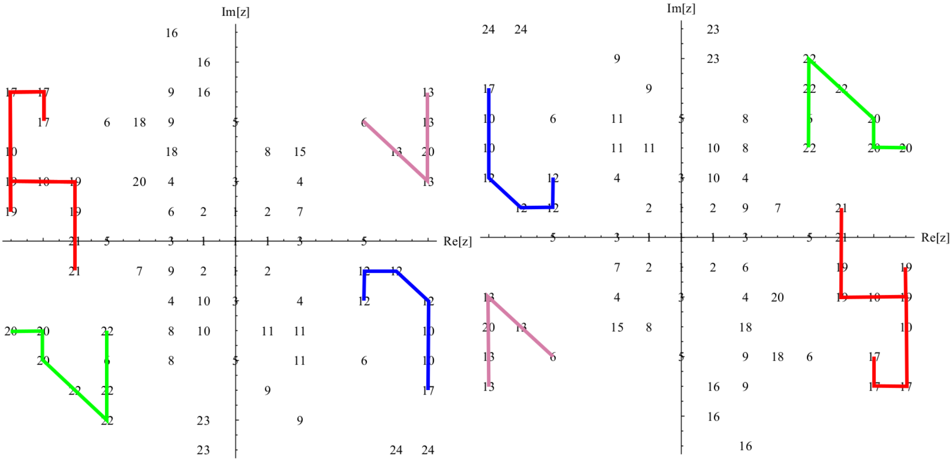
Some Interesting Phenomena for $u = 1$

For 23,730 (out of 40,400) Gaussian integers within $\pm 100 \pm 100i$:

- approximately 5.6% of Gaussian integers land within 10,000 steps;
- the slowest (longest trajectory) Gaussian integer took 251 steps (a tie between $-91 - 98i$ and $-91 - 99i$);
- no Gaussian integer tested took between 252 and 10,000 steps to land on a unit;
- several instances of neighboring Gaussian integers landing in the same number of steps.

If we follow Crandall's argument for the Collatz function over the Gaussian integers, we expect the behavior of the trajectories to increase without bound.

Trajectories up to length 24 using $u = 1$ and $u = -1$



T-K-C Proposition

A Gaussian integer z lands using the unit u_1 if and only if its associate $u_2 z$ lands using the unit $u_1 u_2$.

In particular, z lands on a unit u in n steps using the unit u_1 if and only if its associate $u_2 z$ lands on the unit $u_2 u$ using $u_1 u_2$ in n steps.

Example:

- $10 + 2i$ lands on 1 using the unit 1 in 11 steps;
- $-10 - 2i$ lands on -1 using the unit -1 in 11 steps;
- $-2 + 10i$ lands on i using the unit i in 11 steps;
- $2 - 10i$ lands on $-i$ using the unit $-i$ in 11 steps;

Primary Odd Integer

A **primary odd integer** is an odd Gaussian integer that maps to a power of $1 + i$ under the Gaussian Collatz function, i.e. $C_{2+i,1}(P_n) = (1 + i)^{2n}$.

The set of all primary odd integers are the elements of the sequence $\{P_n\}_1^\infty$ defined recursively by

$$P_1 = i, \quad P_n = P_{n-1} + 2^{n-1}\, i^n$$

Example:

- $P_1 = i \Rightarrow C_{2+i,1}(i) = 2i = (1 + i)^2$
- $P_2 = -2 + i \Rightarrow C_{2+i,1}(-2 + i) = -4 = (1 + i)^4$
- $P_3 = -2 - 3i \Rightarrow C_{2+i,1}(-2 - 3i) = -8i = (1 + i)^6$
- $P_4 = 6 - 3i \Rightarrow C_{2+i,1}(6 - 3i) = 16 = (1 + i)^8$

Scary Form for $\{P_n\}_{n=1}^{\infty}$

We can also express the primary odd integer sequence as

$$P_n = (-1)^{\left\lceil \frac{n-1}{2} \right\rceil} 2a_{\left\lceil \frac{n-1}{2} \right\rceil} + (-1)^{\left\lceil \frac{n}{2} \right\rceil + 1} a_{\left\lceil \frac{n}{2} \right\rceil} i$$

where the a_i come from

$$\{a_m\}_{m=0}^{\infty} = \{0, 1, 3, 13, 51, 205, 819, \ldots\}$$

The recursive sequence is defined by

$$a_0 = 0, \quad a_1 = 1, \qquad a_m = 3a_{m-1} + 4a_{m-2}$$

This sequence is found in other areas of mathematics, see Online Encyclopedia of Integer Sequences (oeis.org).

Coalescing (Merging) Points

Consider the Collatz function over **N**. A **coalescing point** is an integer n that has two pre-images, ie, there exist even x and odd y such that $C(x) = C(y) = n$.

For example, $C(3) = C(20) = 10$. Notice n must be even.

We seek to characterize Gaussian integers that have two pre-images under the Collatz function.

A **coalescing point** z is a Gaussian integer such that it has an even and an odd Gaussian pre-image under the Gaussian Collatz function.

It can be shown that, under $C_{2+i,1}$, a coalescing point $z = a + bi$ must be even and $a \equiv 2b + 1 \bmod 5$.

Coalescing (Merging) Points

The only pairs (a, b) that satisfy the last congruence are:

$$(0, 2), (1, 0), (2, 3), (3, 1), (4, 4)$$

Let z be any Gaussian integer z, and consider the sequence (cycle) of even pre-images $C^{-k}(z) = z(1 + i)^k$ modulo 5. It can be shown that these cycles partition $\mathbb{Z}_5 \times \mathbb{Z}_5$. These elements fall into three of the eight subsets.

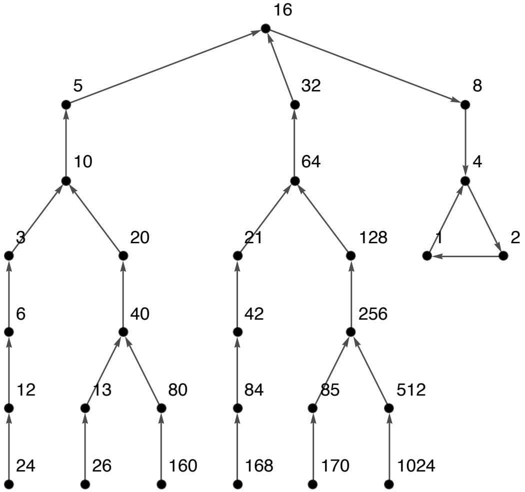
For example $z = 2i$, the even and odd pre-images modulo 5 are $3 + 2i$ and i, respectively. Also, $(0, 2)$ and $(1, 0)$ are in the same subset.

Binary Interpretation in $\mathbb{Z}$

If P_n is odd and maps to a coalescing point of the form 2^{2k}, then $P_n = \sum_{j=0}^{n-1}(2)^{2j}$

Example:

$$P_4 = \sum_{n=0}^{3} 2^{2n}$$
$$= 1 + 2^2 + 2^4 + 2^6 = 85$$



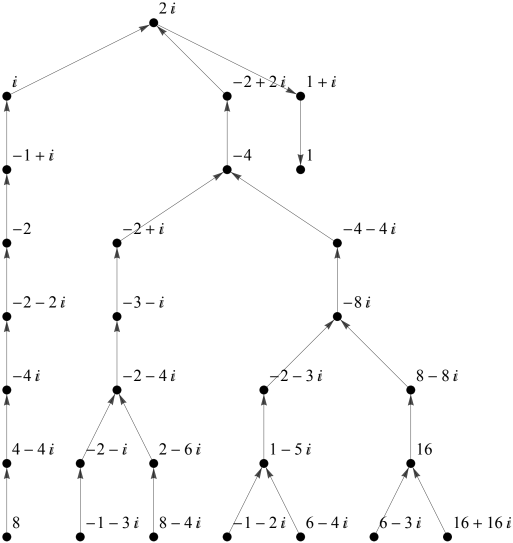
Binary Interpretation in $\mathbb{Z}[i]$

If P_n is odd and maps to a coalescing point of the form $(1+i)^{2k}$,

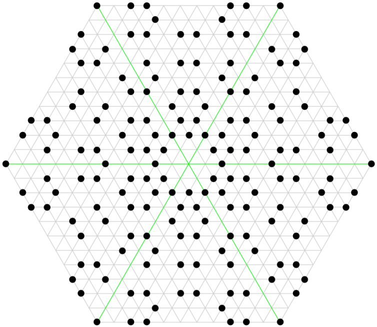
$$P_n = i\sum_{j=0}^{n-1}(i+1)^{2n}$$
$$= i\Big((i+1)^0 + (i+1)^2 + \cdots + (i+1)^{2(n-1)}\Big)$$

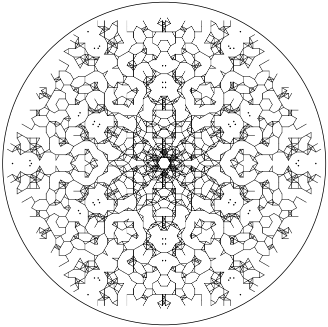
Example:

$$P_4 = i\sum_{n=0}^{3}(i+1)^{2n}$$
$$= i(-3-6i) = 6-3i$$



Motivation: Eisenstein Moat Problem





An Exposition of the Eisenstein Integers, Bandara.

A "walk to infinity" on the Eisenstein primes.

Eisenstein Integers

Can we extend the Collatz function to other rings of integers?

- The Eisenstein integers $\mathbb{Z}[\omega] = \{a + b\,\omega \,|\, \omega^2 + \omega + 1 = 0\}$ forms a triangular lattice in the complex plane.
- The norm $N(a + b\,\omega) = a^2 + b^2 - ab$. The units are $\{\pm 1, \pm\omega, \pm\omega^2\}$.
- If $p \equiv 2 \bmod 3$, then p or $p\,\omega$ are Eisenstein primes. Also, $a + b\omega$ are Eisenstein primes if $N(a + b\,\omega) = p$ where $p = 3$ or $p \equiv 1 \bmod 3$.
- It can be shown that the Eisenstein integers can be factored uniquely as a product of primes (up to associates). For example, $3 = (1 - \omega)(1 - \omega^2)$.
- We were unable to come up with a spiral ordering.

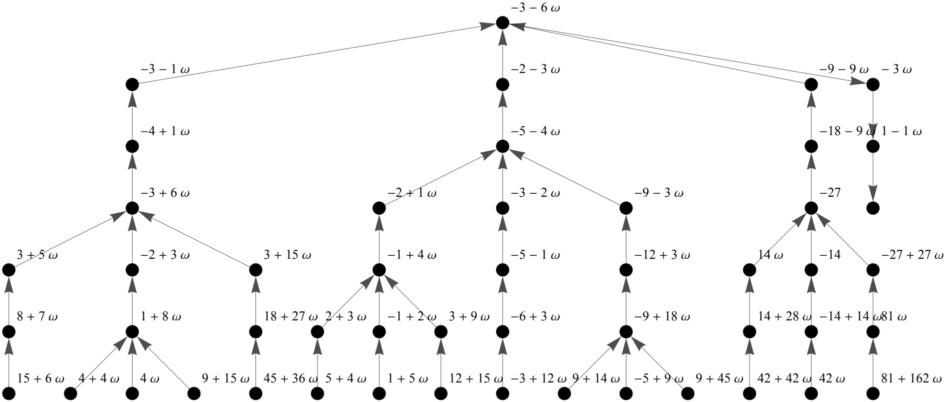
Eisenstein Collatz mapping

Kionke (2016) studied the Collatz mapping over the ring of integers of any algebraic number field $K \neq \mathbb{Q}$. This lead to us searching for appropriate Eisenstein integers to use for our Collatz mapping.

$$E(\alpha) = \begin{cases} \dfrac{(2+2\omega)\alpha+1}{1-\omega} & \text{if } \alpha \equiv 2 \bmod (1-\omega), \\[2ex] \dfrac{2\alpha+1}{1-\omega} & \text{if } \alpha \equiv 1 \bmod (1-\omega), \\[2ex] \dfrac{\alpha}{1-\omega} & \text{if } \alpha \equiv 0 \bmod (1-\omega). \end{cases}$$

Notice three congruence classes modulo $1-\omega$, an Eisenstein prime, and whose norm is 3. If we follow Crandall's argument, it appears that the Eisenstein integers will eventually reach a unit.

Eisenstein Collatz Graph



Future Direction

Thank you!

Paper submitted to Minnesota Journal of Undergraduate Mathematics.

Continue investigating the Collatz Conjecture over other lattices.

"You can get as close as you want to the Collatz conjecture, but it's still out of reach", Tao.

References

- Bandara, S., An exposition of the Eisenstein integers, Masters Theses (2016)
- Crandall, R. E., On the $3x + 1$ problem, Mathematics of Computation **32** (1978)
- Gethner, E., Wagon, S., Wick, B., A Stroll Through the Gaussian Primes, American Mathematical Monthly **105** (1998)
- Gullerud, E. and Mbirika, A., An Euler phi function for the Eisenstein integers and some applications, Integers **20** (2020)
- Hartnett, Mathematician Proves Huge Result on 'Dangerous' Problem, Quanta Magazine (2019)
- Kionke, S. A geometric approach to divergent points of higher dimensional Collatz mappings. Monatsh Math **182** (2017)
- Klee, S. and Tou, E., personal communication (2017)
- Lagarias, J. C., The ultimate challenge: The $3x + 1$ problem, American Mathematical Society (2010)
- Matthews, K. R., Generalized $3x + 1$ mappings: Markov chains and ergodic theory, American Mathematical Society (2010)